

The Sine-Gordon Wobble

G. Kälbermann*

Soil and Water dept., Faculty of Agriculture, Rehovot 76100, Israel

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Abstract

Nonperturbative, oscillatory, winding number one solutions of the Sine-Gordon equation are presented and studied numerically. We call these nonperturbative shape modes *wobble* solitons. Perturbed Sine-Gordon kinks are found to decay to *wobble* solitons.

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*e-mail address: hope@vms.huji.ac.il

1 Introduction

The Sine-Gordon equation was discovered in the study of constant negative curvature metric spaces at the end of the 19th century[1]. It reappeared in physical problems dealing with one dimensional dislocations [2], long Josephson junctions and in other settings.[3]

The Sine-Gordon equation possesses solitary wave solutions. These solitary waves are solitons[4]. The equation is completely integrable and has an infinite number of conserved currents.[2, 5]

The solitons of the Sine-Gordon theory carry a topological winding number q . Sine-Gordon solitary waves are topological solitons. The winding number zero sector $q=0$ supports bound soliton-antisoliton solutions, the breathers, as well as an unbound soliton-antisoliton pairs[2]. The $q=1$ sector solitary wave is the kink soliton.

In the recent past, a controversy has arisen concerning the existence of oscillatory solutions in the $q=1$ sector. Shape modes were predicted by Rice[6] by means of a collective coordinate method. Boesch and Willis[7] studied the excitation of this internal quasimode using a more refined collective coordinate approach as well as numerical integration. The predictions of both works are not exact, or even approximately so, due to the limitations of the collective coordinate method. In the numerical and collective coordinate treatments, the angular frequency of the oscillations of the kink soliton width was found to be above the threshold for the production of phonons. Phonons are the solutions of the Klein-Gordon equation, obtained by linearizing the Sine-Gordon equation[8]. Hence, if this oscillation exists, it is embedded in the continuum and must decay, albeit with a small decay constant for angular frequencies near threshold.

Quintero et al.[9] have recently contested the existence of this shape mode. They have argued that the numerical solution of Boesch and Willis[7] is incorrect. Quintero et al.[9] suggest that this quasi-mode is nothing more than a numerical effect due to discretization. In discrete nonlinear equations, a mode in the continuum, can sink below threshold depending on the value of lattice constant[10].

In the present work we show analytical nonperturbative solutions to the Sine-Gordon equation that are oscillatory and apparently stable that we call *the wobble* solitons. The wobble solitons are derived by means of the Inverse Scattering Transform (IST) method following Lamb[2] and Segur[11]. The IST method produces soliton solutions based on scattering data of Schrödinger-like equations. The data leads to a potential - hence the name inverse scattering transform - from which the soliton is derived. The expression for the wobble soliton we derive corrects the one given by Segur[11]. The solution will be depicted and checked analytically as well as numerically.

The angular frequency of the oscillation of the wobble is found to range between zero and one, where the phonon continuum takes over. There is no gap in the frequency spectrum. A dense set of nonperturbative nonlinear wobbles fills it. We also touch upon the stability issue.

Having shown the existence and probable stability of the wobble, we connect to the problem of shape oscillations in distorted kinks. We recover the results of Boesch and

Willis[7] and point out a probable source of error in the numerical calculations of Quintero et al.[9] The shape modes found in the literature are shown to be an intermediate stage on the way between distorted kinks and wobbles.

The next section summarizes the IST derivation of the the wobble[2],[11] and addresses the stability issue. Section 3 deals with the distorted kink problem and the controversy around the existence of a kink shape mode in the Sine-Gordon equation. Conclusions are presented in section 4.

2 The wobble by the inverse scattering method

The powerful technique of the Inverse Scattering Transform[2], connects between nonlinear equations, such as the Korteweg-deVries (KdV), Sine-Gordon, nonlinear Schrödinger, modified KdV, etc., and linear eigenvalue equations, such as the Schrödinger or Dirac-like two-component equations. The nonlinear equations arise as consistency conditions on the linear equations. The potentials of the linear equations yield solutions of the nonlinear equation. The method uses the scattering data of the linear problem to predict the nonlinear solution by resorting to an integral equation discovered by Gelfand, Levitan and Marchenko[12].

In the case of reflectionless potentials, for which there are only transmitted waves in the linear problem, the Gelfand-Levitan-Marchenko equations are integrable in closed form. The analytical formulae are given by Lamb [2]. Segur[11] implemented these formulae for the case of what we presently call *the wobble*. The Sine-Gordon soliton composed of a kink and a breather, the *wobble* $u(x, t)$ with its center at rest, is given by

$$u(x, t) = 4 \operatorname{Im}(\ln(\det(I + iM))), \quad (1)$$

where $I_{i,j}$, $i, j = 1, 3$ is the unit matrix, and $M_{i,j}$ is the matrix containing scattering data of the kink and the breather.

$$\begin{aligned} M_{j,k} &= \frac{-i m_k}{\zeta_j + \zeta_k} e^{\theta}, \\ \theta &= -i (\zeta_j + \zeta_k) \frac{x+t}{2} + i \frac{x-t}{4 \zeta_k}, \end{aligned} \quad (2)$$

where $\zeta_1 = i/2$, $\zeta_2 = \alpha + i\beta$, $\zeta_3 = -\alpha + i\beta$, with $\alpha^2 + \beta^2 = 0.25$ and $\beta > 0$ are the scattering amplitudes: ζ_1 for the $q=1$ kink and $\zeta_{2,3}$ for the $q=0$ breather. m_j are the normalization constants for each matrix element, with m_1 real and $m_2^* = m_3$. The wobble depends on the parameters $m_2 = (m_{2R}, m_{2I})$, m_1 , β . A moving soliton can be obtained by boosting with a Lorentz transformation. We here focus on a *wobble* at rest.

Defining the complex function

$$F = \det(I + iM) = V + iW, \quad (3)$$

we find ¹

$$\begin{aligned} V(x, t) &= 1 + \frac{|m_2|^2 (\frac{1}{4} - \beta^2)}{\beta^2} e^{4\beta x} - \frac{2 m_1 |m_2| (\frac{1}{2} - \beta)}{\frac{1}{2} + \beta} e^{x+2\beta x} \cos(2\alpha(t + t_0)), \\ W(x, t) &= -\frac{m_1 |m_2|^2 (\frac{1}{2} - \beta)^3}{\beta^2 (\frac{1}{2} + \beta)} e^{x+4\beta x} - m_1 e^x + 2 |m_2| e^{2\beta x} \cos(2\alpha(t + t_0)), \end{aligned} \quad (4)$$

where

$$\frac{2}{|m_2|} \tan(2\alpha t_0) = \frac{m_{2I}\beta + m_{2R}\alpha}{m_{2I}\alpha - m_{2R}\beta}. \quad (5)$$

The *wobble* is obtained by inserting eq.(4) in eq.(1).

As shown below, the *wobble* oscillates sweeping over values above 2π . In numerical codes that limit the inverse tangent to the principal branch, it is imperative to use the complex natural logarithm expression of eq.(1), instead of the translation $\text{Imag}(\ln(F)) = \tan^{-1}(\frac{W}{V})$.

The $q=1$ kink is recovered from the wobble of eqs.(1,4) by setting $m_2 = (0, 0)$. The $q=0$ breather requires the substitution $m_1 = 0$.

The normalization constant m_1 determines the location of the center of the wobble, m_2 fixes the amplitude of the oscillation and the phase. The angular frequency of the oscillation is $\omega = 2\alpha$, with upper bound $\omega_{max} = 1$, at which the the phonon spectrum begins[13],[8].

An alternative simpler form of $F = V + i W$ is

$$\begin{aligned} V(x, t) &= 2 e^{2\beta x} \left(e^\mu \cosh(2\beta x + \mu) - |m_2| e^{x+\lambda-\mu} \cos(2\alpha(t + t_0)) \right), \\ W(x, t) &= 2 e^{2\beta x} \left(-m_1 e^{x+\lambda} \cosh(2\beta x + \lambda) + |m_2| \cos(2\alpha(t + t_0)) \right), \end{aligned} \quad (6)$$

where $e^\lambda = \frac{|m_2|}{|\beta|} \sqrt{\frac{(\frac{1}{2} - \beta)^3}{\frac{1}{2} + \beta}}$, $e^\mu = \frac{|m_2|}{|\beta|} \sqrt{\frac{1}{4} - \beta^2}$.

It is an arduous but straightforward task to show that the wobble obeys the renormalized Sine-Gordon equation.²

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = 0, \quad (7)$$

derivable from the renormalized lagrangian

$$\mathcal{L} = \int dx \left[\left(\frac{\partial u}{\partial t} \right)^2 - \left(\frac{\partial u}{\partial x} \right)^2 + (\cos(u) - 1) \right], \quad (8)$$

¹Eq.(4) corrects the results of Segur[11]

²We have verified eq.(6) using computerized algebra.

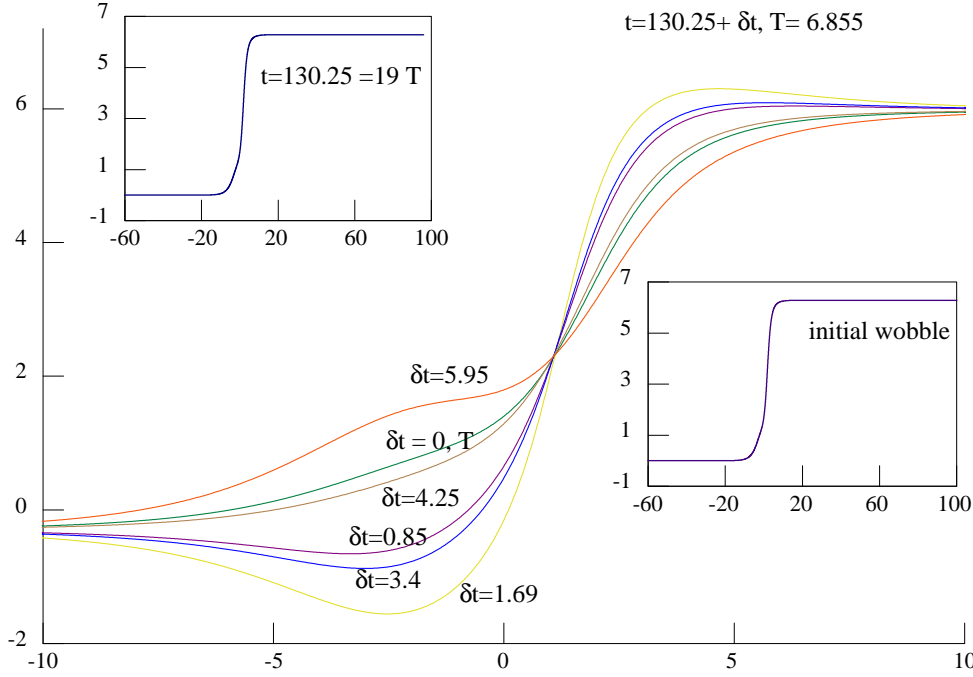


Figure 1: Wobble as a function of distance: $m_1 = -1, m_2 = (0.6, 0.7), \beta = 0.2$

with renormalized energy

$$E = 8 + 32\beta. \quad (9)$$

The remarkable IST method has yielded a new nonlinear solution of the Sine-Gordon equation built from a $q=1$ kink and a $q=0$ breather, a nonperturbative oscillatory shape mode of the kink.

Figure 1 shows snapshots of the wobble at times comprising one oscillation period. As eq.(6) implies, the curves show oscillations that are not simple harmonic. The insets in figure 1 show the wobble at $t=0$ and an identical picture after 19 periods.

The numerical integration code we used is based on the leapfrog method. The calculation used double precision variables with a fixed time step of $dt = 0.01$ and a flexible space grid. One measure of the integration accuracy is the conservation of the energy. The error in the energy was demanded never to exceed 0.2%. The other measure consisted in an exact match with the analytical formulae of eq.(6) for long times. Figure 2 shows one such a comparison for the wobble parameters of figure 1 at $t=137$ amounting to 13700 iterations of the numerical code. The ordinate is the absolute value of the percentage relative deviation of the numerical results $u(x, t)_{num}$ from the analytical formula of eq.(1) $u(x, t)_{form}$, $\epsilon = 100 \text{ abs}\left(\frac{u(x, t)_{num} - u(x, t)_{form}}{u(x, t)_{form}}\right)$. The abscissa spans the region where the results are

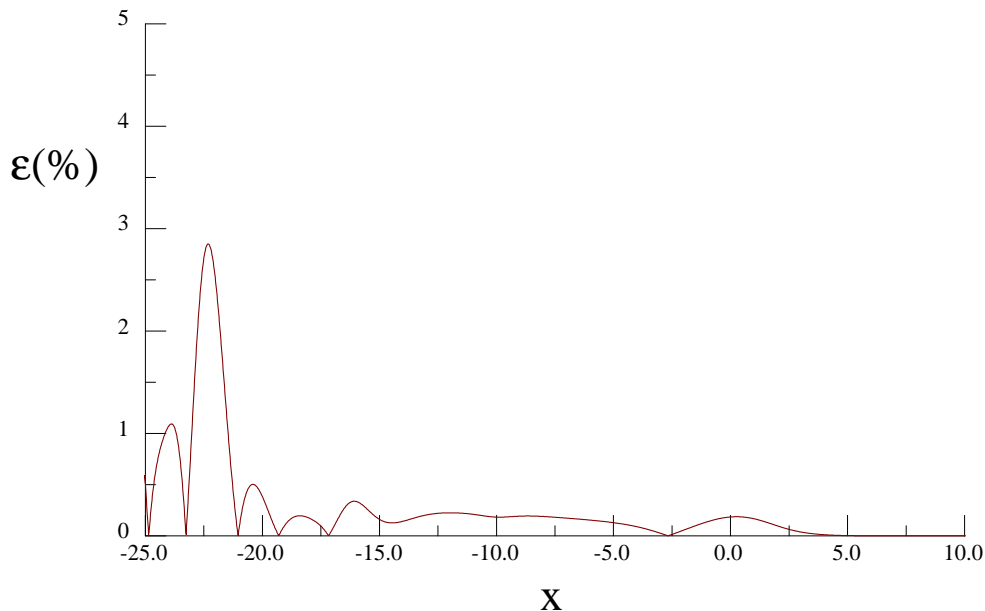


Figure 2: ϵ as a function of distance for the wobble parameters of figure 1

relevant. Below $x=25$ the wobble is negligible and the comparison is irrelevant. The errors very rarely exceed one percent.

The stability problem of distorted solitons under large perturbations has not been settled yet, even for the case of the breather[14]. The study of wobble stability can be circumscribed to the analysis of its development for α, β violating the unitarity condition $\alpha^2 + \beta^2 = \frac{1}{4}$. These parameters belong to the breather sector of the wobble.

Initially we addressed the stability problem by using a limited set of collective coordinates, promoting the parameters to be time dependent. Unfortunately, the method failed to predict the observed behavior. Even the frequency of the unperturbed breather or wobble cannot be recovered by means of the collective coordinate approach.

We therefore proceeded to investigate the question of stability of both the breather and the wobble by means of numerical simulations. We scanned the parameter ranges $-\infty < \alpha < \infty$, $-0.5 < \beta < 0.5$, omitting the $|\beta| > 0.5$ region, for which the distorted breather and wobble decay by emission of soliton-antisoliton pairs.

Figure 3 shows a typical case for the breather, and figure 4 one for the wobble. After a transient that depends on the magnitude of the distortion, both distorted breathers and wobbles eventually settle down at a nearby, lower energy, stable breather or wobble. The excess energy is emitted by means of a trail of phonons both in the forward and backward directions, as seen in the insets of figures 3 and 4. The trail of phonons resembles an Airy function. The pictures in figures 3 and 4 repeat themselves for longer and longer times. The

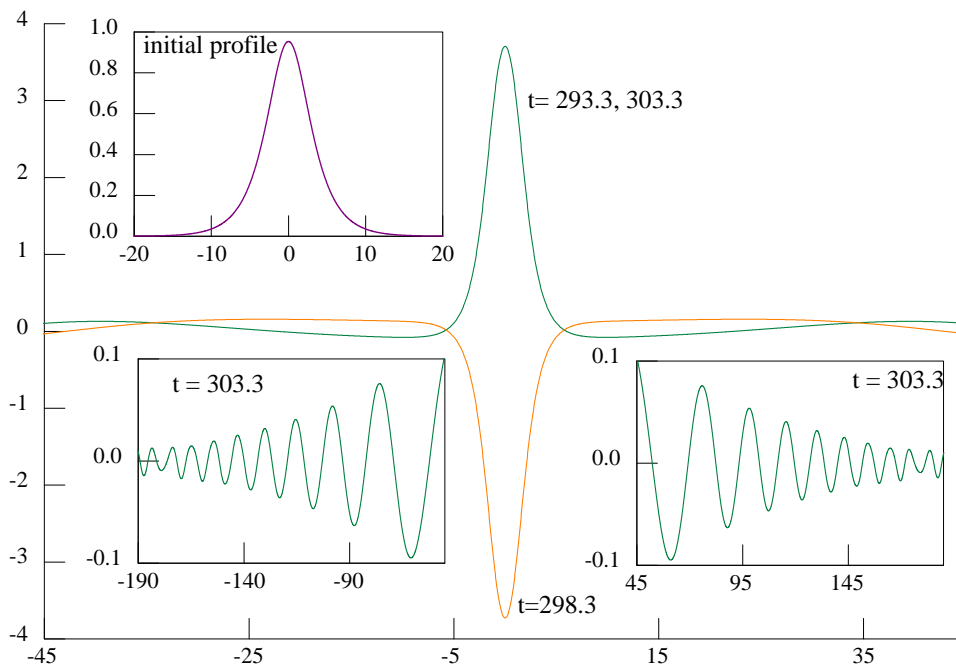


Figure 3: *Long time behavior of an initially distorted breather: $m_1 = 0, m_2 = (0.6, 0.7), \beta = 0.2, \alpha(t = 0) = 0.858, \alpha(t \rightarrow \infty) = 0.314$*

relaxation time was found to be of the order of $\tau = \frac{1}{|\alpha(t = 0) - \alpha(t \rightarrow \infty)|}$. For $t < \tau$, the parameters of the wobble and the breather change with time. At around $t = \tau$, α and β obey again the unitarity constraint and the wobble and breather do not decay appreciably any more. It appears that $\alpha \rightarrow \infty$ depends only on the breather dynamics.

3 Distorted kinks and the Wobble

The existence of a shape mode in the $q=1$ sector of the Sine-Gordon equation has been surrounded by controversy. The results of the previous section show that there are nonperturbative oscillatory shape modes in the $q=1$ kink sector. The wobble angular frequency spectrum fills the gap between the zero mode and the phonon spectrum. The wobble is an exact solution, whereas the phonon spectrum results from approximate linearized solutions of the Sine-Gordon equation around the kink soliton. The wobble must play a role in the decay dynamics of distorted kinks. As we will see below, the shape mode found in the literature, is an intermediate stage on the way from the distorted kink to a stable wobble.

Rice[6], and later Boesch and Willis[7] proposed the existence of shape modes for distorted kinks in the Sine-Gordon equation. The angular frequency of the oscillations of the kink soliton width found by Rice[6] and Boesch and Willis[7] lies above the phonon threshold of

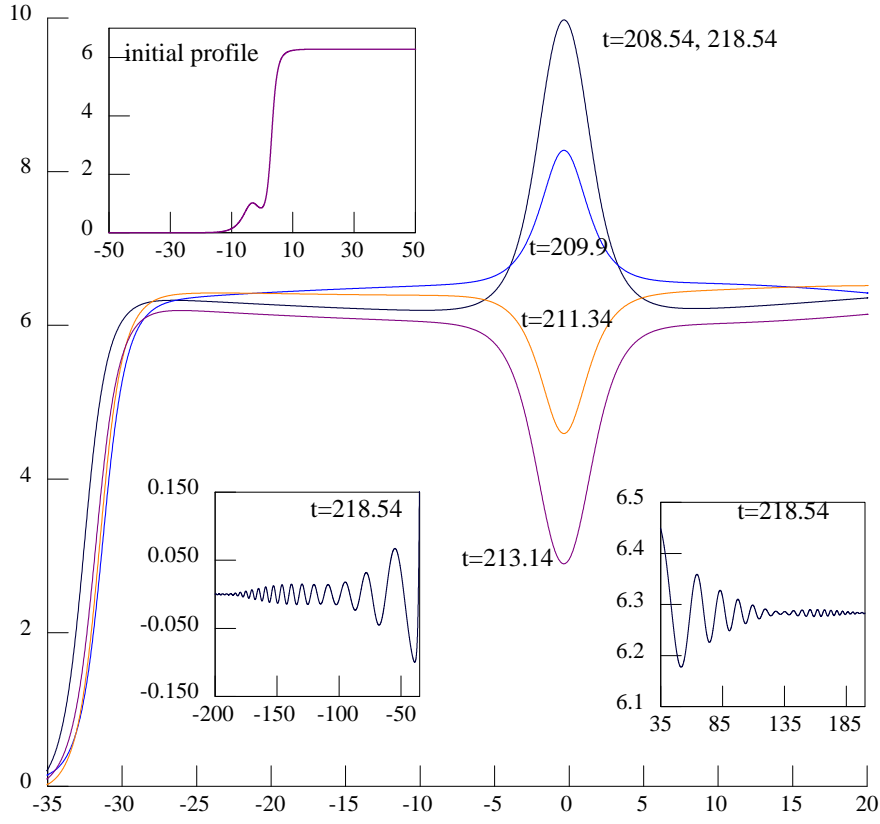


Figure 4: *Long time behavior of an initially distorted wobble* : $m_1 = 1, m_2 = (0.6, 0.7), \beta = 0.2, \alpha(t = 0) = 0.858, \alpha(t \rightarrow \infty) = 0.314$

$\omega = 1$. The shape mode is therefore unstable to decay into phonons. Recently, Quintero et al.[9] have argued that such a shape mode does not exist. Quintero et al.[9] claim that the behavior observed in the work of Boesch and Willis[7] is due to discretization effects in the numerical calculation. They base their hypothesis on a work by Kivshar et al.[10]. Kivshar et al.[10] predict the birth of shape modes bifurcating from the continuum spectrum of phonons that plunge below threshold upon the application of perturbations. For the unperturbed integrable Sine-Gordon equation the effect disappears in the continuum, when lattice spacing tends to zero. Kivshar et al.[10] state that there are no shape modes for integrable equations. The results of the previous section show that there is a whole set of nonperturbative shape modes in the Sine-Gordon equation.

In order to connect the wobble soliton to perturbed kinks, we followed the evolution of a distorted kink

$$u(x, t) = 4 \tan^{-1}(e^{\gamma x}), \quad (10)$$

with $|\gamma - 1|$ the distortion parameter. $\gamma(t)$ is extracted from the data by comparing to eq.(10). The phenomenological function

$$\gamma(t) = 1 + a e^{-t^b c} \cos(2\alpha t + \phi), \quad (11)$$

captures the broad features of $\gamma(t)$. From eq.(11) we obtained the angular frequency of the oscillation, α . We found that α varies with time. There is a slow drift of α towards lower values. This is depicted in figure 5. The angular frequency α of eq.(11) drops from $2\alpha = 1.048$ at around $t=200$ to $2\alpha = 1.0015$ later. Both the amplitude of the oscillation and the frequency diminish gradually. The results of figure 5 agree in general with Boesch and Willis[7]. However, contrarily to the predictions of the collective coordinate method of Rice[6], the frequency is not constant.

We consider now the numerical simulations of Quintero et al.[9]. Quintero et al.[9] use an insufficient extent for the x axis $L = |x_{max}| = 100$, that does not prevent the reabsorption of reflected phonons from the boundary. These phonons pump back energy into the oscillating soliton and blur the picture. The velocity of propagation of the phonons is $v = \frac{k}{\sqrt{k^2 + 1}}$, asymptotically tending towards $v = 1$. Using this asymptotic value, the reflected phonons collide and feed energy back to the soliton at $t \approx 200$. In figure 2 of [9], and at approximately that time, the decaying single frequency shape mode picture starts to break down.

Increasing the span of the integration region with time prevents the reabsorption of phonons. The slowly drifting single frequency behavior is seen to persist for longer and longer times. Whether the oscillation frequencies cross the threshold of $\omega = 1$, signaling the transition to a stable shape mode looks unclear from the previous figures. To accelerate the decay process a very distorted kink is needed. Figure 6 shows a case with $\gamma(t = 0) = 0.4$, a distortion of 60% compared to the unperturbed kink. (The distorted kink energy for this γ is $E = 11.6$ still below the threshold for the production of a soliton-antisoliton pair at $E = 24$).

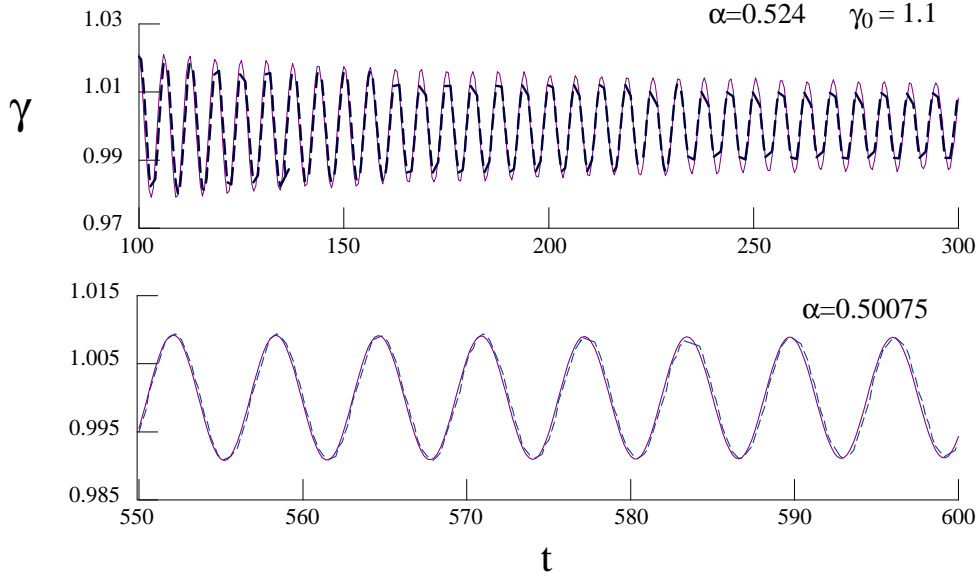


Figure 5: *Distorted kink width γ of eq.(10) as a function of time. Fitted values of eq.(11), full curve; input data, dashed curve*

The angular frequency is now $2\alpha = 0.97$ for short times. A long wavelength modulation of the amplitude is also noticeable in figure 6. After $t=600$, the amplitude of the oscillation appears to stabilize.

As α was obtained by means of a phenomenological function, more convincing evidence of the transition to a wobble-like regime is necessary.

A distorted kink cannot be put in exact correspondence with the wobble, despite the similarities. The energy of eq.(9) teaches us that β is the relevant parameter for the breather admixture to the kink. Expanding the expression for the wobble of eq.(1) around $\beta = 0$ we find $\gamma(t=0) \approx 1 - 8\beta$. The factor of 8, and the unitarity constraint that fixes the angular frequency to be $\alpha = \sqrt{\frac{1}{4} - \beta^2}$ require an extremely distorted kink in order to reach a fairly visible frequency below threshold. A stronger distortion than $\gamma(t=0) = 0.4$ as compared to that depicted in figure 6 is necessary.

We therefore considered initial distortions with $\gamma(t=0) < 0.4$. For such a large initial distortion, the decaying kink profile did not match eq.(10) any more. The extraction of a clean distortion parameter $\gamma(t)$ became impossible. The time evolution of the highly distorted kinks leads to a completely different object. Figure 7 shows distorted kinks for $\gamma(t=0) = 0.3$ at around $t = 290$. The profiles resemble very much the wobbles figure 1.

From the graphs one can read off the value of the angular frequency of the oscillation to be $2\alpha \approx 0.92$, well below the phonon threshold. The inset shows phonons receding from the center, others propagating forwards, are not shown. We performed long time numerical

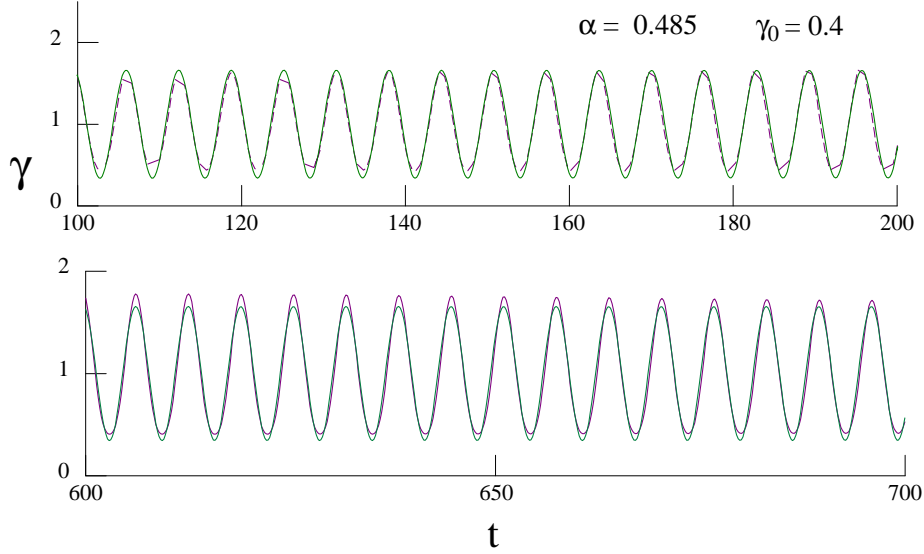


Figure 6: *Distorted kink width γ of eq.kink as a function of time, full curve, fitted values, dashed curve, numerical input*

integrations up to $t=1000$ and did not see any sizeable decay of the amplitude of the wobble-like kink.

The kink appears to be decaying to a wobble. If this is indeed the case, we should be able to identify the wobble parameters of the decaying distorted kink.

To limit the number of free parameters, we first considered the $x \rightarrow \infty$ region for both the distorted kink and the wobble. The wobble has an asymptotic behavior of $e^{2|\beta|x}$, whereas for the distorted kink it is $e^{\gamma x}$. If the asymptotic behavior does not change with time, we have $\gamma \approx 2\beta$. From this value of β and the unitarity constraint, α is determined. This $\alpha = \frac{\pi}{T}$ with T the oscillation period, can be readily compared to the numerical results of figure 7. The agreement is fair, but not satisfactory. Following the reverse path seemed more appropriate. The oscillation period of the numerical data fixes α and consequently β by means of the unitarity constraint. There remained three unknown parameters m_1 , m_{2R} , m_{2I} that were determined using a minimization algorithm.

The results are depicted in figure 8. All the distorted kinks of figure 7 were reproduced with the same parameter set. The agreement with the data is remarkable, especially so in light of the highly nonlinear wobble function. Long distance discrepancies are due to phonon contributions.

The highly distorted kink has transformed into a wobble and a wake of phonons. The less distorted cases presumably need a much longer time to reach a wobble. For small distortions of the kink, it is hard to discern a clear wobble shape. However, we cannot rule out completely the possibility of a distorted kink decaying to an undistorted kink and phonons. In future work we plan to address this and other related problems.

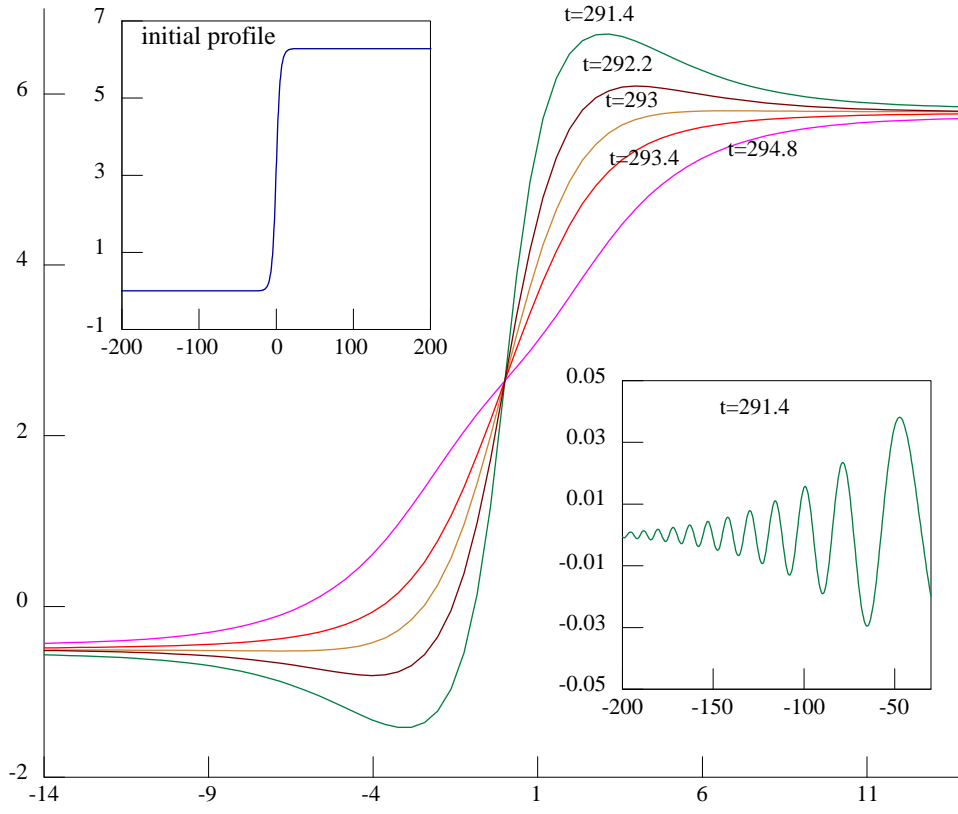


Figure 7: *Distorted kinks for $\gamma(t = 0) = 0.3$, as a function of distance, at various times, showing oscillatory behavior*

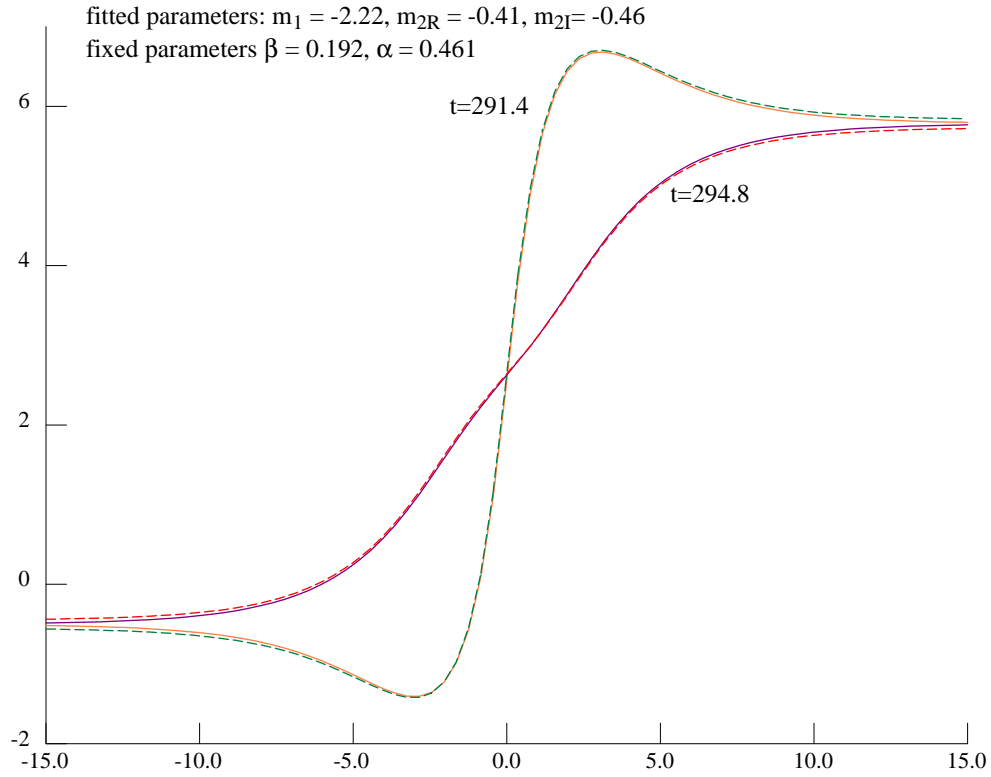


Figure 8: *Distorted kinks at $t=291.4$, $t=294.8$ for $\gamma(t=0) = 0.3$, as a function of distance. Numerical data of figure 7, broken line; wobble fit, full line.*

4 *Conclusions*

We have shown that there exists a set of wobbling, apparently stable, nonperturbative solutions to the Sine-Gordon equation in the $q=1$ sector. Highly distorted kinks eventually decay to wobbles and phonons. The results are relevant to the investigation of scattering events of Sine-Gordon kinks from impurities. Other nonlinear equations that support breathers, such as the modified KdV equation, may bear wobble solutions too.

The existence of the wobble may have technological implications. Wobbles produced in Josephson junctions could carry analogical information with relative stability.

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